On Casey's Inequality

Casey's theorem is a famous result in geometry (see [3;4]). Ptolemy's theorem (see [2]), can be viewed as a special case of Casey's theorem. Ptolemy's inequality (see [3]), on the other hand, can be considered as an extension of Ptolemy's theorem. In this article we will prove an extension of Ptolemy's inequality.

Theorem 1 (Casey's theorem)**.** *Circles c_1, c_2, c_3, c_4 are tangent to a fifth circle or a straight line if and only if*

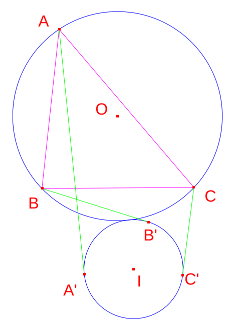
$$T_{(12)}T_{(34)} \pm T_{(13)}T_{(42)} \pm T_{(14)}T_{(23)} = 0,$$

where $T_{(ij)}$ is the length of a common tangent to circles i and j.

Theorem 2. *Let ABC be a triangle inscribed in circle (O). Circle (I) touches (O) at a point on the arc $\overset{\frown}{BC}$ which does not contain A. From A, B, C draw tangents AA', BB', CC' to (I). Prove that*

$$aAA' = bBB' + bCC',$$

where a, b, c are the sides of triangle ABC.



Theorem 3 (Casey's inequality)**.** *Let ABC be a triangle inscribed in the circle (O) and let (I) be an arbitrary circle. From A, B, C draw tangents AA', BB', CC' to (I). Prove that*

1. *If $(I) \cap (O) = \varnothing$ then $a \cdot AA', b \cdot BB', c \cdot CC'$ are sidelengths of a triangle.*
2. *If $(I) \cap (O) \neq \varnothing$: say the point of intersection lies on*
   - *arc $\overset{\frown}{BC}$ which does not contain A, then $aAA' \geq bBB' + cCC'$*
   - *arc $\overset{\frown}{CA}$ which does not contain B, then $bBB' \geq cCC' + aAA'$*
   - *arc $\overset{\frown}{AB}$ which does not contain C, then $cCC' \geq aAA' + bBB'$.*

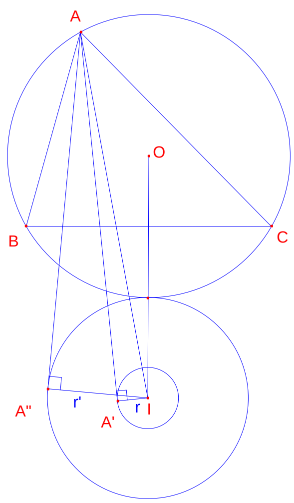
Equality holds if and only if circle (I) is tangent to (O).

Proof. 1. Let $(I) \cap (O) = \varnothing$ and let r be the radius of circle (I). Draw a circle (I, r') (concentric with circle I and radius r') which touches (O) at a point on the arc $\overset{\frown}{BC}$ which does not contain A. It is not diffucult to see that $r' \geq r$. Draw tangents AA'', BB'', CC'' to (I, r'), where $A'', B'', C'' \in (I, r')$, respectively. By the Pythagorean theorem we have

$$AA'^2 + r^2 = IA^2, AA''^2 + r'^2 = IA^2.$$

Therefore $AA'^2 = AA''^2 + r'^2 - r^2$ and, analogously,

$$BB'^2 = BB''^2 + r'^2 - r^2,\ CC'^2 = CC''^2 + r'^2 - r^2. \tag{1}$$



From Theorem 2, squaring both sides yields

$$a^2 AA''^2 = b^2 BB''^2 + c^2 CC''^2 + 2bcBB''CC''. \tag{2}$$

Now if we prove that $bBB' + cCC' \geq aAA' \geq |bBB' - cCC'|$ then $a \cdot AA', b \cdot BB', c \cdot CC'$ will be sidelengths of a triangle. Indeed, the inequality $bBB' + cCC' \geq aAA'$ is equivalent to

$$\begin{aligned}
b^2BB'^2 + c^2CC'^2 + 2bcBB'CC' &\geq a^2AA'^2 \\
b^2(BB''^2 + r'^2 - r^2) + c^2(CC''^2 + r'^2 - r^2) + 2bcBB'CC' &\geq a^2AA'^2 && \text{From (1)} \\
(b^2 + c^2 - a^2)(r'^2 - r^2) - 2bcBB''CC'' + 2bcBB'CC' &\geq 0 && \text{From (2)} \\
2bc\cos A(r'^2 - r^2) - 2bcBB''CC'' + 2bc\sqrt{(BB''^2 + r'^2 - r^2)(CC''^2 + r'^2 - r^2)} &\geq 0 && \text{From (1)} \\
\cos A(r'^2 - r^2) - BB''CC'' + \sqrt{(BB''^2 + r'^2 - r^2)(CC''^2 + r'^2 - r^2)} &\geq 0.
\end{aligned}$$

By the Cauchy-Schwarz inequality

$$\sqrt{(BB''^2 + r'^2 - r^2)(CC''^2 + r'^2 - r^2)} \geq BB''CC'' + r'^2 - r^2.$$

Note that the inequality $\cos A(r'^2 - r^2) + r'^2 - r^2 \geq 0$ is true because $r' \geq r$ and $(1 + \cos A) \geq 0$. This completes the proof.

Now the inequality $aAA' \geq |bBB' - cCC'|$ is equivalent to

$$b^2BB'^2 + c^2CC'^2 - 2bbBB'CC' \leq a^2AA'^2.$$

Like above, we are left to prove that

$$\cos A(r'^2 - r^2) - BB''CC'' - \sqrt{(BB''^2 + r'^2 - r^2)(CC''^2 + r'^2 - r^2)} \geq 0.$$

Because $-\sqrt{(BB''^2 + r'^2 - r^2)(CC''^2 + r'^2 - r^2)} \leq -BB''CC'' - (r'^2 - r^2)$, the left-hand side is less than or equal to

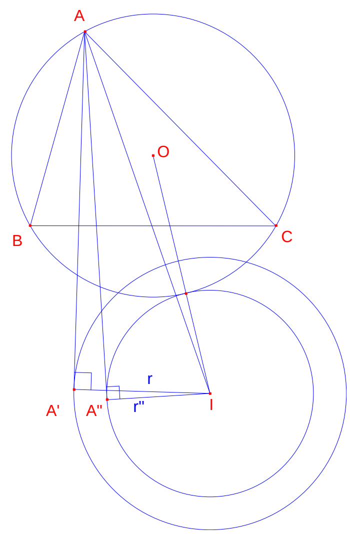
$$\cos A(r'^2 - r^2) - r'^2 - r^2 - 2BB''CC'' < 0.$$

2. Let $(I) \cap (O) \neq \varnothing$ and (I, r) intersects arc $\overset{\frown}{BC}$ which does not contain A. Draw (I, r'') which touches arc $\overset{\frown}{BC}$ that does not contain A. It is not difficult to see that $r'' \leq r$. Draw tangents AA'', BB'', CC'' to (I, r''), where $A'', B'', C'' \in (I, r'')$, respectively. By the Pythagorean theorem, as in (1), we get

$$AA'^2 = AA''^2 + r''^2 - r^2, BB'^2 = BB''^2 + r''^2 - r^2, CC'^2 = CC''^2 + r''^2 - r^2$$

or

$$AA''^2 = AA'^2 + r^2 - r''^2, BB''^2 = BB'^2 + r^2 - r''^2, CC''^2 = CC'^2 + r^2 - r''^2. \quad (3)$$



Use Theorem 2 and (3) to get the form

$$\cos A(r''^2 - r^2) - BB''CC'' + BB'CC' \leq 0. \quad (4)$$

Note that

$$BB''CC'' = \sqrt{(BB'^2 + r^2 - r''^2)(CC'^2 + r^2 - r''^2)} \geq BB'CC' + r^2 - r''^2$$

and the left-hand side is less than or equal to

$$\cos A(r''^2 - r^2) - (r^2 - r''^2) = (r''^2 - r^2)(1 + \cos A) \leq 0,$$

because $r'' \leq r$ and $1 + \cos A \geq 0$. □

References

[1] http://mathworld.wolfram.com/PtolemyInequality.html

[2] http://mathworld.wolfram.com/PtolemysTheorem.html

[3] Roger A. Johnson, *Advanced Euclidean Geometry* Dover Publications (August 31, 2007)

[4] http://mathworld.wolfram.com/CaseysTheorem.html

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